Exercise 7.2

Superposition in spherically symmetric spacetimes We explained in the text that Einstein's equations are nonlinear and solutions cannot be superposed. A surprising exception to this result is in the case of spherically symmetric spacetimes of a particular kind. This exercise explores this feature.

(a) Consider a metric of the form

$$ds^2 = -f(r)dt^2 + f^{-1}(r)dr^2 + r^2 d\Omega^2 \tag{7.39}$$

with a general function f(r) that needs to be determined via Einstein's equations. Show that this metric will satisfy Einstein's equations provided the the source energy-momentum tensor has the form

$$T^t_t = T^r_r = -\frac{\epsilon(r)}{8\pi G}; \quad T^\theta_\theta = T^\phi_\phi = -\frac{\mu(r)}{8\pi G} \tag{7.40}$$

Equation (7.40) also defines the functions $\epsilon(r)$ and $\mu(r)$.

Einstein's equation is

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G T_{\mu\nu} \tag{1}$$

where R is the Ricci scalar. We need to solve this equation for the (7.39) metric. Surely, one new to general relativity must solve Einstein's equation by writing it down. But here as routine as it is, we use the MATHEMATICA (with xAct Package) to solve Einstein's equation for the metric (7.39). The final result is

$$E^\nu{}_\mu = \begin{pmatrix} \frac{1}{r^2}\left(rf'(r)+f(r)\right) & 0 & 0 & 0 \\ 0 & \frac{1}{r^2}\left(rf'(r)+f(r)\right) & 0 & 0 \\ 0 & 0 & \frac{1}{2r}\left(rf''(r)+2f'(r)\right) & 0 \\ 0 & 0 & 0 & \frac{1}{2r}\left(rf''(r)+2f'(r)\right) \end{pmatrix} \tag{2}$$

As can be seen $E^r_r = E^r_r$ and $E^\theta_\theta = E^\phi_\phi$ and can be generalized as functions $\epsilon(r)$ and $\mu(r)$ and (7.40) can be achieved.

(b) Show that the Einstein equations now reduce to:

$$\frac{1}{r^2}(1-f) - \frac{f'}{r} = \epsilon; \qquad \nabla^2 f = -2\mu \tag{7.41}$$

The remarkable feature about the metric in (7.39) is that Einstein's equations become linear in $f(r)$ so that solutions for different $\epsilon(r)$ can be superposed.

Both can be seen in (2).

(c) Given any $\epsilon(r)$, integrate the Eq. (7.41) to determine the solution to be

$$f(r) = 1 - \frac{a}{r} - \frac{1}{r}\int_a^r \epsilon(r)r^2 dr \tag{7.42}$$

with a being an integration constant chosen such that $f = 0$ at $r = a$ and $\mu(r)$ is fixed by $\epsilon(r)$ through with a being an integration constant chosen such that $f = 0$ at $r = a$ and $\mu(r)$ is fixed by $\epsilon(r)$ through

$$\mu(r) = \epsilon + \frac{1}{2}r\epsilon'(r) \tag{7.43}$$

Starting from the left equation in (7.41), we are dealing with a linear nonhomogenous differential equation.

$$f' + \frac{f}{r} = \frac{1}{r} - r\epsilon(r) \tag{3}$$

The solution of the general equation

$$f' + p(r)f = q(r) \qquad \text{Solution:} \qquad f(r) = v(r)e^{P(r)} + Ae^{P(r)}$$

where $v'(r) = e^{-P(r)}q(r)$ and $P(r)$ is antiderivative of $-p(r)$ and A arised from initial condition. Therefore

$$P(r) = \ln r \quad \Rightarrow \quad v'(r) = e^{\ln r}\left(\frac{1}{r} - r\epsilon(r)\right) \quad \Rightarrow \quad v(r) = r + a - \int r^2\epsilon(r)$$

and consequently

$$f(r) = 1 + \frac{a}{r} - \frac{1}{r}\int r^2\epsilon(r) \tag{4}$$

fixing the integration constant a to make sure $f(r)$ is zero where $r = a$, we get (7.42). To get (7.43), using E^θ_θ

$$\frac{1}{2r}\left(rf''(r) + 2f'(r)\right) \tag{5}$$

Now taking a derivative of (3)

$$f'' = -\frac{1}{r^2} - \frac{f'}{r} + \frac{f}{r^2} - \epsilon(r) - r\epsilon'(r) \tag{6}$$

Putting the above equation and derivative of (3) in (5)

$$\begin{aligned}
&\frac{1}{2r}\left(-\frac{1}{r} - f' + \frac{f}{r} - r\epsilon(r) - r^2\epsilon'(r) + 2f'(r)\right)\\
&= \frac{1}{2}\left(-\frac{1}{r^2} + \frac{f}{r^2} + f' - \epsilon(r) - r\epsilon'(r)\right)\\
&= \frac{1}{2}\left(-2\epsilon(r) - r\epsilon'(r)\right)
\end{aligned}$$

where in the third line we have used (3). Therefore

$$\mu(r) = \epsilon(r) + \frac{1}{2}r\epsilon'(r) \tag{7}$$